of a turbulent stream for large wave numbers in [3], we assumed, in fact, that the dispersions of the straining rate tensor in the Lagrangian and Eulerian descriptions were the same. We have now proved this fact.

A simple transformation of (4) yields the following expression for the relative motion of fluid particles in a homogeneous stream:

$$\int W_L(t, \mathbf{r}, \mathbf{u} \mid \mathbf{r}_0)\, d^3 r_0 = W_E(t, \mathbf{r}, \mathbf{u}) \tag{11}$$

Here W_L is the combined density of the distributions of the velocity difference and of the distance between fluid particles initially separated by the distance $\mathbf{r}_0$; W_E is the density of the velocity difference distribution at two fixed points the distance $\mathbf{r}$ apart. Formula (4) can also be used to obtain several other new relations.

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ON THE STATIC THEORY OF TWO-DIMENSIONAL TURBULENCE

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In contrast to three-dimensional motions, two-dimensional motions have not only the usual energy integral, but also an integral of motion which is quadratic in the velocity, namely the square of the curl of the velocity field. As is shown in [1, 2], this fact ensures the existence of a solution of the hydrodynamics equations with a normal (Gaussian) distribution of the velocity field probabilities with a spectrum different from white noise.

Our purpose in the present paper is to determine the characteristic of such a distribution, i. e. the correlation (structural) function of the fields under investigation, directly from the hydrodynamics equations.

Let us consider the two-dimensional motion of an incompressible inviscid turbulent fluid in the xy-plane. We assume that the turbulence is stationary in time and homogeneous and isotropic in space. The motion of the fluid is described by the stream function $\psi(\mathbf{r}, t)$ which satisfies the equation

$$\frac{\partial}{\partial t}\Delta\psi = \{\Delta\psi, \psi\}, \qquad \{\varphi, \psi\} = \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x} \tag{1}$$

Here $\{\varphi, \psi\}$ are the Poisson brackets and Δ is the two-dimensional Laplacian. The velocity field is defined by the vector $(-\partial\psi / \partial y,\ \partial\psi / \partial x)$.

In accordance with the results of [1, 2] we assume that the field of the stream function $\psi(\mathbf{r}, t)$ is random and normally distributed. Averaging Eq. (1) over the ensemble of turbulent motions, we obtain

$$\langle\{\Delta\Psi, \Psi\}\rangle = 0 \tag{2}$$

Since the field $\Psi(\mathbf{r}, t)$ of the stream function has been determined to within an additive constant, the statistical characteristic of the field Ψ is its structural function

$$D_\psi(|\mathbf{r}_1 - \mathbf{r}_2|) = \langle[\Psi(\mathbf{r}_1 t) - \Psi(\mathbf{r}_2, t)]^2\rangle \tag{3}$$

Let us consider the product of three ψ-functions taken at a single instant but at different points in space. By virtue of stationarity,

$$\partial / \partial t \langle\Delta\Psi(\mathbf{r}_1, t)\, \Delta\Psi(\mathbf{r}_2, t)\, \Delta\Psi(\mathbf{r}_3, t)\rangle = 0 \tag{4}$$

Making use of Eqs. (1), (3), we readily obtain a functional equation for the structural function D_ψ, i.e.

$$X_{q_1, q_2} + X_{q_2, q_3} + X_{q_3, q_1} = 0 \tag{5}$$

$$X_{q_1, q_2} = \frac{1}{q_1 q_2} \frac{\partial^2}{\partial q_1 \partial q_2} [\Delta_{q_1}{}^2 D_\psi(q_1)\, \Delta_{q_2} D_\psi(q_2) - \Delta_{q_2}{}^2 D_\psi(q_2)\, \Delta_{q_1} D_\psi(q_1)] \tag{6}$$

Here $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ ($q_i = |\mathbf{q}_i|$) are the vectors defined by the relations

$$\mathbf{q}_1 = \mathbf{r}_1 - \mathbf{r}_2, \qquad \mathbf{q}_2 = \mathbf{r}_2 - \mathbf{r}_3, \qquad \mathbf{q}_3 = \mathbf{r}_3 - \mathbf{r}_1 \qquad (\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0)$$

In deriving Eq. (5) we made use of the normal character of the distribution of the field Ψ (the average of the product of four Ψ-functions is equal to the sum of the averages of all possible pairs of Ψ-functions).

Expression (6) implies that

$$X_{q_1, q_2} = - X_{q_2, q_1} \tag{7}$$

Condition (7) enables us to solve functional equation (5). Its solution is of the form

$$X_{q_1, q_2} = A(q_1) - A(q_2) \tag{8}$$

Here $A(q)$ is an arbitrary function. Multiplying (8) by $q_1 q_2$ and applying the operator $\partial^4 / \partial q_1^2 \partial q_2^2$, we obtain the following differential equation for the structural function $D_\psi(q)$:

$$\frac{\partial^6}{\partial q_1^3 \partial q_2^3} [\Delta_{q_1}{}^2 D_\psi(q_1)\, \Delta_{q_2} D_\psi(q_2) - \Delta_{q_2}{}^2 D_\psi(q_2)\, \Delta_{q_1} D_\psi(q_1)] = 0 \tag{9}$$

We can solve Eq. (9) by separating variables. Making use of the condition $\Delta D_\psi(q) \to 0$ as $q \to \infty$, we obtain

$$(\Delta + \lambda)\Delta D_\psi(q) = 0 \tag{10}$$

Here λ is the separation constant which has the dimensions of the square of inverse length.

In solving Eq. (10) we must deal with the two cases corresponding to the values

$$\lambda > 0 \ (\lambda = k_0^2), \quad \lambda < 0, \ (\lambda = -k_0^2)$$

In the first case the solution of (10) for ΔD_ψ is

$$\Delta D_\psi(q) = C J_0(k_0 q) \tag{11}$$

Here $J_0(z)$ is a Bessel function. The quantity $\Delta D_\psi(q)$ determines the structural function of the velocity field and therefore the spectral energy density, which in our case is given by

$$E(k) = E_0 \delta(k - k_0) \tag{12}$$

This corresponds to a discrete spectrum.

In the second (more interesting) case we obtain the following solution for ΔD_ψ which decays rapidly with increasing q:

$$\Delta D_\psi(q) = - C_1 k_0^2 K_0(q / L_0) \tag{13}$$

where $K_0(z)$ is a Macdonald function and the quantities $L_0 = k_0^{-1}$ and C_1 are dimensional constants which arise in the theory. The spectral energy density corresponding to (13) is of the form

$$E(k) = \frac{E_0 k}{k_0^2 + k^2} \qquad (14)$$

The behavior of this spectral density is characterized by the logarithmic divergence of the average kinetic energy density. This divergence is due to the absence of viscosity in the problem under consideration.

Carrying out bivariate Fourier transformation in formula (13), we obtain the following expression for the bivariate spectrum of the stream function field:

$$D_\psi(k) = \frac{C_2}{k^2(k_0^2 + k^2)} \qquad (15)$$

This coincides with the result of [1]. The expression for the structural function D_ψ itself can be obtained by integrating Eq. (1.3) under the conditions $D_\psi(0) = 0$, $D_\psi(0) = 0$

$$D_\psi(q) = C_1\left[K_0(q/L_0) + \ln\frac{q}{2L_0} + \gamma\right] \qquad (16)$$

where $\gamma \approx 0.58$ is Euler's constant. A distinctive feature of the behavior of $D_\psi(q)$ is the absence of a second derivative at zero. Fig. 1 shows $D_\psi(q)/C_1$ as a function of the dimensionless length $\xi = q/L_0$.

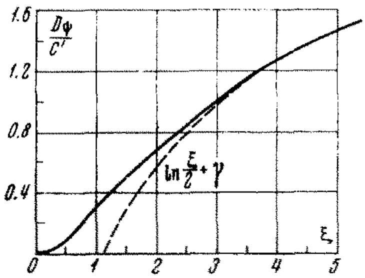

Fig. 1

Our assumption that the field of $\psi(\mathbf{r}, t)$ is distributed normally is not necessary. All of our results remain valid for a quasinormal distribution (the hypothesis of [3]), although the resulting solution may not be unique. We also infer from the results of [4, 5] that the spectral energy density in the analogous three-dimensional problem turns out to be that of white noise, which is of no physical interest.

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